Reassessing D8 Optimization-Analytically

D8: I can use calculus and derivative tests to find extreme values and classify them as maxima or minima.

Basic Preparation

1. Did you do the written practice?
2. Did you do the WeBWorK?
3. Go back to your notes, any handouts from class, the desmos activities, WeBWorK and any written practice you were able to use to prepare. Compare this to your quiz/homework.
4. Do you understand what your mistake was? If so, briefly describe what the mistake is below. If you are unsure, please go to the Calculus Center and work with a tutor until you can describe what your mistake was.

Metacognition

Now, *WHY* did you make the mistake? Answering this question is asking you to think about HOW you think about math (metacognition). Spending time here will help you become more efficient at learning math and is therefore worth the time!

1. Was your incorrect answer due to
   (a) not understanding a concept;
   (b) an error in logical reasoning (e.g., used the correct theorem/test but made the wrong conclusion, used a theorem/test/technique when it did not apply);
   (c) being careless (e.g. not reading directions, not answering the question completely, making arithmetic or basic algebra errors);
   (d) not knowing how to start or formulate an approach to the problem;
   (e) others?

   Briefly describe why your answer was incorrect:

2. What helped you recognize your mistake(s). Here are some examples: the course notes, the textbook, homework or conversations from the Calculus Center. In other words, which strategies for identifying mistakes work well for you and will help you in the future?

3. Rework the ENTIRE PROBLEM. Rewrite your solution from start to finish, carefully fixing the mistake(s) you diagnosed above. By doing the entire problem over again, you can make sure you fix your mistake and better understand the point of the exercise.

4. Describe (in detail) what you have done in order to learn from your mistake(s) and prepare for your next attempt. Did you read the textbook or class notes? Did you look at examples and/or work problems on your own or with your tutor/classmate/instructor, and if so, which problems? Did you take a different approach than listed here? (Again, the point of this isn't just to look at what you did on this problem, but how can you learn from this and be more likely to meet expectations on future assignments on the first try.)

Where topic was first introduced: Module 10

Basic Principles

1. Identify all of the x-values worth investigating.
2. Test those x-values.

Digging Deeper

1. Identify all of the x-values worth investigating.
   - Extrema can only occur at endpoints or critical points.
   - Use the context of the problem to define the endpoints.
   - Critical points are values $x = p$ in the domain of f where $f'(p) = 0$ OR $f'(p)$ is undefined.
   - Direction changes can happen when f' is 0 or undefined, so even if those x values aren't critical points, they must be considered.
2. Test those x-values.
   - Option 1: State that your function is continuous on a closed interval, then compare the output values at the endpoints and critical points. This only works when the function is continuous on a closed interval!
   - Option 2: Use the First Derivative Test to show that the function is changing from increase to decrease or vice versa around the critical point. This only works if you have correctly identified ALL x values where the slope can change and are calculating the derivatives on the correct intervals.
   - Option 3: Use the Second Derivative Test to determine the concavity at a critical point (cp) where $f' = 0$ and conclude if there is a max or min. This only works when $f'(cp) = 0$ and $f''(cp)$ exists and is non-zero.

Digging Even Deeper

1. Critical points: Remember that until you have memorized the definition it is a good idea to write it down from the source.

   Critical point - We say a function, f, has a *critical point* at $x = p$ if p is **in the domain** of f and $f'(p) = 0$ OR $f'(p)$ is undefined. The point $(p, f(p)$ on the graph of f is also called a *critical point.*

   ("in the domain" means that the parent function exists at that point, or that $f(p)$ exists.)

   There are three basic parts to check:
   - Are there any x-values where $f(x)$ does not exist? (domain)
   - When does $f'(x) = 0$?
   - When is $f'(x)$ DNE?

   You must check for BOTH where f' is 0 or undefined, OR STATE why there are no x values where f' is 0 or undefined. Otherwise, once you start testing your results could be flawed.

Critical Points: Favorite Mistakes

- Checking a few specific x-values to see if the derivative is 0 or DNE. This is a problem because the first derivative test uses direction changes to determine if we have a max or min. If we are missing an x value, our test results might be wrong.
- Just because a place where f' DNE is NOT a critical point doesn't mean that something interesting isn't happening there. Test between all places where f' is 0 or undefined.

Example

Let $f'(x) = \dfrac{2(x-1)^3(x-5)}{(x-3)^3}$. $f'(x) = 0$ when $x = 1$ and when $x = 5$. If we were to use only these two points, we would need to know increase/decrease on three intervals: $(-\infty, 1)$, $(1, 5)$, and $(5, \infty)$.

Let's say we chose to test $x = -10$, 2, and 10. When we test $f'(-10)$, we get a negative number. When we test $f'(2)$, we get a positive number. When we test $f'(10)$ we also get a positive number. Thus we would conclude that because f' changes from negative to positive around $x = 1$, $f(1)$ will be a local minimum. We would also conclude that because there is no sign change in f' around $x = 5$, there is no max or min when $x = 5$.

Graph $f(x) = \dfrac{(x-4)^4}{(x-3)^2}$ and zoom out so you can see $y-$values between 0 and 100. What is happening when $x = 5$? How can we be sure to KNOW that we have tested enough intervals? What went wrong in the analysis above?

- Not checking to see if the denominator is zero (this would make $f'(x)$ DNE).

Example

Let $f'(x) = \dfrac{2(x-1)^3(x-5)}{(x-3)^3}$ again. When you have a fraction (rational function) like this, there is a shortcut:

- $f'(x) = 0$ if the numerator $= 0$.
  Set $2(x-1)^3(x-5) = 0$. Because the numerator is factored, we can set each factor equal to zero, so if $(x-1)^3 = 0$, then we can cube root both sides and get $(x-1) = 0$ and finally that $x = 1$. Same with $(x-5) = 0$: solving for x gives $x = 5$. Thus we see that if $x = 1$ or $x - 5$ the numerator is zero, and thus the entire derivative is 0.

- $f'(x)$ DNE if the denominator $= 0$
  Set $(x-3)^3 = 0$. Cube root both sides, get $(x-3) = 0$, solve for x: get $x = 3$. Thus, if $x = 3$, the denominator is zero and the entire derivative is undefined.

Note

By definition, since $f(1)$ and $f(5)$ both exist (plug these numbers into $f(x)$ above and see if you get an answer), we know they are in the domain of f, and because $f' = 0$ at these points, they are both critical points. However, $f(3)$ is not defined, so we can't call $x = 3$ a critical point. BUT!!! we must include $x = 3$ in our test with the first derivative, otherwise we might not test numbers close enough to $x = 5$ to determine accurately that there is a direction change around 5 and that we have a local minimum at $(5, f(5))$.

- Not showing conclusively there are not other points of interest.
  - If you end up trying to take the square root of a negative number while solving for a critical point, this tells you that there is no critical point there.
  - Consider rewriting negative exponents as fractions: for example my mind might look at $x^{-1/2}$ and think, oh, this exists if $x = 0$, but if I rewrite as $\frac{1}{x^{1/2}}$ it's painfully obvious that if $x = 0$ the fraction does NOT exist (can't divide by 0).

Example

Let $f'(x) = 1 + \frac{2}{3}x^{-2/3}$. If we reframe this to make it easier to analyze, we can write $f'(x) = 1 + \dfrac{2}{3x^{2/3}}$. It is easy to see that x is in the denominator so $f'(0)$ is undefined, and since $f(0)$ is defined, we know we have a critical point at $x = 0$. But we can't stop there, because there might be other critical points.

Set $f'(x) = 0$	$1 + \dfrac{2}{3x^{2/3}} = 0.$
Subtract the 1 from both sides:	$\dfrac{2}{3x^{2/3}} = -1.$
Clear the fraction by multiplying both sides by the denominator:	$2 = -3x^{2/3}.$
Divide by -3:	$-\dfrac{2}{3} = x^{2/3}$

Observe that we are trying to find a value for x so that when squared and cube rooted, we will get a *negative* number. Since we have x^2, we can never get a negative number on the right side of the equation, so there is no value for x that will balance this equation.

By writing out this argument, we show there are no places (x-values) where f' can be 0 or DNE other than $x = 0$.

2. Using one of the three tests to determine local/global extrema.
   - Option 1: State that your function is continuous on a closed interval, then compare the output values at the endpoints and critical points. This only works when the function is continuous on a closed interval!
   - Option 2: Use the First Derivative Test to show that the function is changing from increase to decrease or vice versa around the critical point. This only works if you have correctly identified ALL x values where the slope can change and are calculating the derivatives on

the correct intervals.

- Option 3: Use the Second Derivative Test to determine the concavity at a critical point (cp) where $f' = 0$ and conclude if there is a max or min. This only works when $f'(cp) = 0$ and $f''(cp)$ exists and is non-zero.

Testing: Favorite Mistakes

- Skipping this step. If you skip this step, you might have found a minimum instead of a maximum, or maybe the actual max is at an endpoint instead of a critical point etc. You MUST use a test to finish off the problem.

- Not showing enough work: First Derivative Test
  The bare minimum for the first derivative test includes letting us know which x numbers you plugged into the first derivative, and then drawing your conclusions.

- Not showing enough work: Test for Global Extrema
  This test only works if you have a continuous function on a closed interval, thus you must show us that you are working with a continuous function on a closed interval.

- Using the Second Derivative Test like it's the First Derivative Test
  The Second derivative Test works on concavity *AT* the critical point. It is a mistake to test for concavity on either side of the critical point.

  Note that because we test AT the critical point, if you have parameters in your problem, the second derivative might be an easier choice. Otherwise, if you try the first derivative test, you need to find x values larger or smaller than your critical point. For example, if your critical point is $a/2$, a number smaller than $a/2$ is $\frac{a/2}{2} = a/4$, and a number bigger than $a/2$ is $a/2 \cdot 2 = a$. (Basically I cut the critical point in half to get a smaller number, and I doubled it to get a larger number.)

Examples:

(b is a positive constant.)

1. Find the critical points of $f(x) = b\sqrt[4]{x}$.

   $f'(x) = \frac{b}{4}x^{-3/4}$. $f'(0)$ is undefined but 0 is in the domain, so 0 is a critical point. f' never $=0$, so 0 is the only critical point.

2. Find the critical points of $f(x) = x + b\sqrt[5]{x}$.

   $f'(x) = 1 + \frac{b}{5}x^{-4/5}$. $f'(0)$ is undefined but 0 is in the domain, so 0 is a critical point. $f' = 0$ if $x^4 = (-5/b)^5$, but we can't take the 4th root of a negative number so 0 is the only critical point.

3. Find the critical points of $f(x) = x + \frac{b}{x^2}$.

   $f' = 1 - 2bx^{-3} = 1 - \frac{2b}{x^3}$. $f'(0)$ is undefined BUT $f(0)$ is also undefined, so 0 is not in the domain of f thus $x = 0$ cannot be a critical point. $f' = 0$ if $x = \sqrt[3]{2b}$, which is the only critical point.

4. Find the critical points of $f(x) = x - \frac{b}{x^3}$.

   $f' = 1 + 3bx^{-4} = 1 + \frac{3b}{x^4}$. $f'(0)$ is undefined BUT $f(0)$ is also undefined, so 0 is not in the domain of f thus $x = 0$ cannot be a critical point. $f' = 0$ if $x = \sqrt[4]{-3b}$, but we cannot take an even root of an odd number so this is not a critical point.

5. Let $f(x) = x + \frac{b}{x^3}$ Does $x = \sqrt[4]{3b}$ yield a local max or min?

   $f' = 1 - 3bx^{-4} = 1 - \frac{3b}{x^4}$. Note that when $x = 0$ there could be a direction change even though we don't call $x = 0$ a critical point. So, when using the first derivative test, pick a number between 0 and $x = \sqrt[4]{3b}$, and a number larger than $x = \sqrt[4]{3b}$. To find a number smaller, divide by 2. To find a number larger, multiply by 2. To be particularly lazy, divide or multiply before taking the 4th root.

   $$f'(\sqrt[4]{3b/2}) = 1 - \frac{3b}{(\sqrt[4]{3b/2})^4} = 1 - \frac{3b}{3b/2} = 1 - 2 = -1 < 0$$

   $$f'(\sqrt[4]{6b}) = 1 - \frac{3b}{(\sqrt[4]{6b})^4} = 1 - \frac{3b}{6b} = 1 - 1/2 = 1/2 > 0$$

   Because f' switches from negative to positive, there is a local minimum when $x = \sqrt[4]{3b}$ Alternatively, we could have used the second derivative test: $f''(x) = 12bx^{-5}$.

   $$f''(\sqrt[4]{3b}) = 12b(\sqrt[4]{3b})^{-5} > 0$$

   Because the second derivative at the critical point is positive, f is concave up and we have a minimum at the critical point.

6. Use information about first and second derivatives to identify where a function has critical points, and determine if the endpoints and critical points are local extrema.

   Use the information below to classify the critical points and endpoints as local max, min or neither.

   - $f(x)$ is continuous on [-1,5].
   - $f'(x) = 0$ only when $x = 0$, and $x = 2$.
   - $f'(x)$ does not exist only when $x = 1$.
   - $f'(-0.5) < 0$
   - $f'(0.5) < 0$
   - $f'(1.5) > 0$
   - $f''(2) < 0$

   Endpoints are -1 and 5. Critical points are 0, 1 and 2. There are no other points of interest.
   We know that f is decreasing when $x = -0.5$ and $x = 0.5$. Thus, f is decreasing on the intervals $(-1, 0)$ and $(0, 1)$. Because $f'(1.5) > 0$, f is increasing on the interval $(1, 2)$.
   $f''(2) < 0$ tells us that when $x = 2$, f is concave down.
   Consider x=-1. Because the function is decreasing on $(-1, 0)$, there must be a local max when $x = -1$ as function values close to $f(-1)$ must decrease as x increases.
   We see that f is decreasing on both sides of $x = 0$, so there cannot be a local extreme value when $x = 0$.
   Because the function changes from decrease to increase around $x = 1$, there is a local minimum when $x = 1$.
   When $x = 2$ we have $f'(2) = 0$ and $f''(2) < 0$ so we can apply the second derivative test and conclude that there is a local maximum.
   (We can make an argument that there is a local minimum when $x = 5$, but this is tricky! In a quiz setting you will either be given more information or won't be asked about that x value. Can you figure out why there must be a minimum there?)

Prepare for Revision:

Use ALGEBRA and CALCULUS!! (but check work with a graphing utility.)
(c is a positive constant.)

1. Find all critical points and classify them as local max/min/neither: $f(x) = 5x^3 + 2x^2 - 10x + 6$

2. Find all critical points and classify them as local max/min/neither: $g(x) = \dfrac{(x-1)^3}{(x-3)}$

3. Find all critical points and classify them as local max/min/neither: $f(x) = \sqrt[4]{cx}$

4. Find all critical points and classify them as local max/min/neither: $f(x) = x + \sqrt[5]{cx}$

5. Find all critical points and classify them as local max/min/neither: $f(x) = x + \frac{4}{cx^4}$

6. Find all critical points and classify them as local max/min/neither: $f(x) = x - \frac{3}{cx^5}$

7. Let $f(x) = cx + \frac{1}{x^5}$ Does $x = \sqrt[6]{5/c}$ yield a local max or min? Does $x = -\sqrt[6]{5/c}$ yield a local max or min?

8. Use the information below to classify the critical points and endpoints as local max, min or neither.

   - $f(x)$ is continuous on [-5,7].
   - $f'(x) = 0$ only when $x = -3$, $x = 0$ and $x = 2$.
   - $f'(x)$ does not exist only when $x = 4$.
   - $f'(-4.5) < 0$
   - $f'(-2.5) < 0$
   - $f''(0) > 0$
   - $f'(3) > 0$
   - $f'(5) < 0$

   Classify the following x values as max or min or neither: x =-5, -3, 0, 4, 7.